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OPTIMUM MULTIPLE TUNED MASS DAMPERS FOR BASE-EXCITED UNDAMPED SYSTEM

R. S. JANGID*,†
Department of Civil Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

SUMMARY

Optimum parameters of Multiple Tuned Mass Dampers (MTMD) for an undamped system to harmonic base excitation are investigated using a numerical searching technique. The criteria selected for the optimality is the minimization of steady-state displacement response of the main system. The explicit formulae for the optimum parameters of MTMD (i.e. damping ratio, bandwidth and tuning frequency) are then derived using curve-fitting scheme that can readily be used for engineering applications. The error in the proposed explicit expressions is investigated and found to be quite negligible. The optimum parameters of the MTMD system are obtained for different mass ratios and number of dampers. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: optimum parameters; MTMD; bandwidth; harmonic; base-excitation

INTRODUCTION

Tuned Mass Damper (TMD) is a classical engineering device consisting of a mass, a spring and a viscous damper attached to a vibrating main system in order to attenuate any undesirable vibration. The natural frequency of the damper system is tuned to a frequency near to the natural frequency of the main system, the vibration of the main system causes the damper to vibrate in resonance, as a result, the vibration energy is dissipated through the damping in the tuned mass damper. The main disadvantage of a single TMD is its sensitivity of the effectiveness to the error in the natural frequency of the structure and/or that in the damping ratio of the TMD. The effectiveness of a tuned mass damper is reduced significantly by the mistuning or off-optimum damping. As a result, the use of more than one tuned mass dampers with different dynamic characteristics has been proposed in order to improve the effectiveness. Multiple tuned mass dampers with distributed natural frequencies were proposed by Xu and Igusa[1] and also studied by Yamaguchi and Harnpornchai[2] and Jangid and Datta.[3] It is shown that the MTMD is more effective for vibration control as compared to single TMD.

*Correspondence to: R. S. Jangid, Department of Civil Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 40076, India. E-mail: rsjangid@civil.iitb.ernet.in
† Assistant Professor.

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Here-in, optimum damping, bandwidth and tuning frequency of the MTMD system for an undamped main system to harmonic-base excitation is investigated. A numerical searching technique is used for obtaining the optimum parameters for minimization of the steady-state displacement response of the main system. The explicit mathematical expressions for the optimum parameters are then derived using curve-fitting scheme. Further, a comparison is made between the optimum parameters by numerical technique and closed form expressions to investigate the error.

STRUCTURAL MODEL

Consider an undamped main system supported by n number of tuned mass dampers with different dynamic characteristics as shown in Figure 1. The main system is characterized by the stiffness k_s and the mass m_s. The parameters of the jth tuned mass damper are mass m_j, damping c_j, and stiffness k_j. Natural frequencies of the MTMD are uniformly distributed around their average frequency. The natural frequency, ω_j (i.e. $\sqrt{k_j/m_j}$) of the jth TMD is expressed by

$$\omega_j = \omega_T \left[1 + \left(j - \frac{n+1}{2} \right) \frac{\beta}{n-1} \right] \tag{1}$$

and

$$\omega_T = \sum_{j=1}^{n} \omega_j / n \tag{2}$$

$$\beta = \frac{\omega_n - \omega_1}{\omega_T} \tag{3}$$

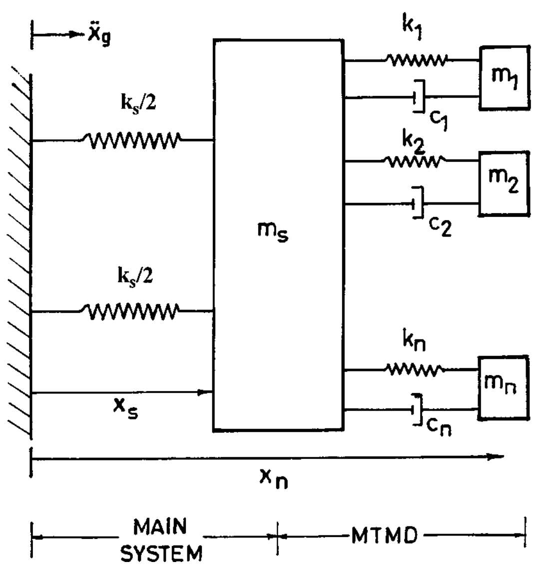

Figure 1. Structural model of a main system with MTMD

where ω_T is the average frequency of all MTMD and β is the non-dimensional frequency bandwidth of the MTMD system.

As suggested by Xu and Igusa[1] that the manufacturing of the MTMD with uniform stiffness is simpler as compared to that with varying stiffness. As a result, the distribution of natural frequencies of the MTMD is made by keeping the stiffness constant and varying the mass (i.e. $k_1 = k_2 = \cdots k_n = k_T$).

The damping constant of the jth TMD is expressed as

$$c_j = 2m_j \xi_T \omega_j \tag{4}$$

where ξ_T is the damping ratio kept constant for all MTMD.

Total mass of the MTMD system is expressed by the mass ratio defined as

$$\mu = \frac{\sum_{j=1}^{n} m_j}{m_s} \tag{5}$$

where μ is the mass ratio of the MTMD system.

Tuning frequency ratio of the MTMD system is expressed as

$$f = \frac{\omega_T}{\omega_s} \tag{6}$$

where ω_s is the natural frequency of the main system.

RESPONSE TO HARMONIC-BASE ACCLERATION

The main system is excited by the harmonic-base acceleration given by

$$\ddot{x}_g = \ddot{x}_0 e^{i\omega t} \tag{7}$$

where $\ddot{x}_g$ is the base acceleration, $\ddot{x}_0$ is the amplitude, ω is the circular frequency and $i = \sqrt{-1}$.

Amplitude of steady-state harmonic displacement of the main system, $x_s(\omega)$ to $\ddot{x}_g$ is expressed by[1]

$$x_s(\omega) = \frac{m_s - (i\omega)^{-1} Z(\omega)}{k_s - \omega^2 m_s - i\omega Z(\omega)} \ddot{x}_0 \tag{8}$$

where

$$Z(\omega) = -i\omega \sum_{j=1}^{n} \frac{m_j(k_j - i\omega c_j)}{k_j - i\omega c_j - \omega^2 m_j} \tag{9}$$

The amplitude of the main system is expressed in the normalized form as

$$R = \frac{\omega_s^2 |x_s(\omega)|}{\ddot{x}_0} \tag{10}$$

EVALUATION OF OPTIMUM PARAMETERS

In Figure 2 variation of R against harmonic excitation frequency is shown for single TMD and MTMD system. The response R is plotted for four different damping ratios (i.e. $\xi_T = 0{\cdot}01$, $0{\cdot}02$,

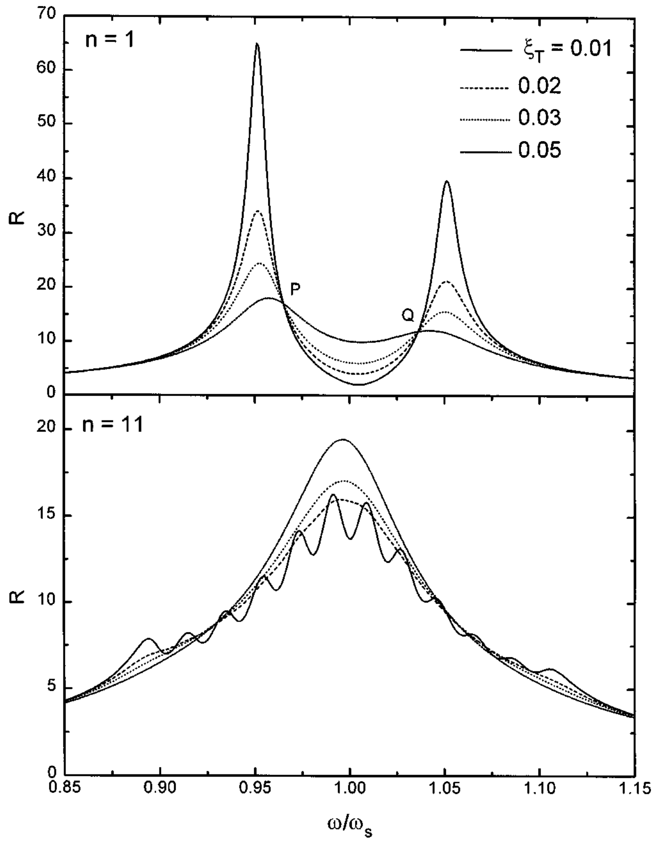

Figure 2. Variation of response amplitude R against harmonic excitation frequency for different damping ratio of the tuned mass dampers ($\mu = 0{\cdot}01$, $\beta = 0{\cdot}2$ and $f = 1$)

0·03 and 0·05) with other parameters as $\mu = 0{\cdot}01$, $\beta = 0{\cdot}2$ and $f = 1$. Figure indicates that for single TMD system the curve for R passes through two fixed points (shown as P and Q) for all damping ratios of the TMD. Den Hartog[4] described the method for finding out the optimum parameters of a single TMD using these two fixed points. Subsequently, Warburton[5] obtained the closed-form expression for optimum parameters of single TMD for different combination of response and excitation. Note that for the MTMD system there does not exist such common points for which the R intersects for all damping ratios. As a result, the closed-form expressions for optimum parameters of the MTMD system cannot be obtained using the same procedure of the single TMD. However, the value of the optimum parameters of the MTMD system can be determined by numerical searching technique. Tsai and Lin[6] has used similar procedure for finding out the optimum parameters of single TMD for the damped main system where similar situation exists.

For a given mass ratio (μ) and number of MTMD (n), the parameters of the MTMD (i.e. ξ_T, β and f) are varied such that the displacement amplitude R of the main system attains the minimum value for all harmonic excitation frequencies. The constraints applied on the values of

parameters for the optimization study are: $0 \leqslant \xi_T < 1$; $0 \leqslant \beta < 2$ and $f > 0$. These conditions satisfy (i) the natural frequencies of the MTMD are positive real and (ii) the MTMD are under-damped. The superscript opt is used to denote the optimum parameters and corresponding displacement amplitude R.

Based on the above numerical searching technique the optimum parameters of MTMD system and corresponding displacement amplitude of the main system for different number of MTMD and the mass ratios are obtained and shown in Tables I and II. Using these optimum parameters,

Table I. Variation of optimum parameters against the number of MTMD

n	$\mu = 0{\cdot}01$				$\mu = 0{\cdot}05$				$\mu = 0{\cdot}1$			
	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}
1	0·0611	0·0000	0·98762	14·2861	0·1352	0·0000	0·94040	6·6460	0·1890	0·0000	0·88606	4·9270
3	0·0315	0·0861	0·99104	12·0260	0·0696	0·1861	0·95668	5·6297	0·0975	0·2548	0·91693	4·1860
5	0·0240	0·1113	0·99202	11·5141	0·0526	0·2424	0·96134	5·3812	0·0727	0·3345	0·92573	3·9976
7	0·0201	0·1239	0·99247	11·3411	0·0443	0·2700	0·96350	5·2920	0·0607	0·3738	0·92983	3·9271
9	0·0183	0·1307	0·99270	11·2712	0·0393	0·2867	0·96476	5·2542	0·0585	0·3887	0·93144	3·9005
11	0·0166	0·1360	0·99288	11·2421	0·0365	0·2970	0·96551	5·2376	0·0548	0·4021	0·93272	3·8902
15	0·0161	0·1401	0·99296	11·2246	0·0350	0·3065	0·96600	5·2293	0·0520	0·4162	0·93377	3·8821
21	0·0156	0·1436	0·99301	11·2175	0·0340	0·3140	0·96626	5·2255	0·0513	0·4254	0·93403	3·8792
31	0·0155	0·1460	0·99302	11·2133	0·0331	0·3204	0·96647	5·2232	0·0497	0·4347	0·93450	3·8767

Table II. Variation of optimum parameters against the mass ratio of MTMD

μ	$n = 5$				$n = 11$				$n = 21$			
	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}
0·005	0·0172	0·0788	0·99599	16·1860	0·0122	0·0958	0·99640	15·8144	0·0111	0·1018	0·99649	15·7801
0·010	0·0240	0·1113	0·99202	11·5141	0·0166	0·1360	0·99288	11·2421	0·0156	0·1436	0·99301	11·2175
0·015	0·0294	0·1357	0·98807	9·4566	0·0205	0·1657	0·98935	9·2275	0·0189	0·1756	0·98958	9·2069
0·020	0·0338	0·1562	0·98416	8·2368	0·0239	0·1903	0·98582	8·0333	0·0218	0·2021	0·98617	8·0148
0·025	0·0375	0·1743	0·98029	7·4086	0·0263	0·2127	0·98238	7·2227	0·0243	0·2253	0·98278	7·2056
0·030	0·0409	0·1904	0·97644	6·8005	0·0285	0·2327	0·97898	6·6273	0·0263	0·2465	0·97947	6·6116
0·035	0·0443	0·2047	0·97261	6·3303	0·0306	0·2508	0·97559	6·1669	0·0293	0·2639	0·97597	6·1525
0·040	0·0472	0·2182	0·96883	5·9533	0·0331	0·2666	0·97215	5·7978	0·0307	0·2822	0·97276	5·7842
0·045	0·0499	0·2308	0·96507	5·6427	0·0349	0·2822	0·96881	5·4936	0·0326	0·2983	0·96945	5·4808
0·050	0·0526	0·2424	0·96134	5·3812	0·0365	0·2970	0·96551	5·2376	0·0340	0·3140	0·96626	5·2255
0·055	0·0551	0·2534	0·95764	5·1574	0·0389	0·3095	0·96208	5·0188	0·0362	0·3274	0·96290	5·0071
0·060	0·0570	0·2645	0·95402	4·9631	0·0409	0·3218	0·95873	4·8289	0·0381	0·3404	0·95960	4·8177
0·065	0·0599	0·2736	0·95031	4·7926	0·0431	0·3330	0·95537	4·6625	0·0400	0·3526	0·95632	4·6515
0·070	0·0615	0·2839	0·94676	4·6415	0·0449	0·3442	0·95207	4·5152	0·0415	0·3648	0·95313	4·5042
0·075	0·0639	0·2926	0·94314	4·5064	0·0467	0·3548	0·94879	4·3837	0·0440	0·3747	0·94971	4·3733
0·080	0·0656	0·3018	0·93963	4·3848	0·0478	0·3660	0·94565	4·2655	0·0458	0·3852	0·94647	4·2552
0·085	0·0671	0·3109	0·93615	4·2746	0·0502	0·3746	0·94227	4·1585	0·0475	0·3954	0·94324	4·1484
0·090	0·0698	0·3179	0·93256	4·1742	0·0514	0·3847	0·93914	4·0610	0·0492	0·4051	0·94004	4·0512
0·095	0·0712	0·3264	0·92915	4·0822	0·0531	0·3936	0·93592	3·9720	0·0504	0·4152	0·93699	3·9617
0·100	0·0727	0·3345	0·92573	3·9976	0·0548	0·4021	0·93272	3·8902	0·0513	0·4254	0·93403	3·8792

a curve-fitting scheme is carried out to find out the closed-form expressions of the optimum parameters in terms of μ and n of the MTMD system. After several trials and errors following expressions are found for the optimum parameters that provide the minimum error:

$$\xi_T^{\mathrm{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)(1-0.5\mu)}} + (a_1 + a_2\sqrt{\mu} + a_3\mu)\sqrt{\mu}\left\{a_4\left(\frac{1}{\sqrt{n}} - 1\right) + a_5\left(\frac{1}{n} - 1\right) + a_6(\sqrt{n} - 1)\right\} \tag{11}$$

$$\beta^{\mathrm{opt}} = (a_1 + a_2\sqrt{\mu} + a_3\mu)\sqrt{\mu}\left\{a_4\left(\frac{1}{\sqrt{n}} - 1\right) + a_5(n-1) + a_6(\sqrt{n} - 1)\right\}\frac{1}{\sqrt{n}} \tag{12}$$

$$f^{\mathrm{opt}} = \frac{\sqrt{1 - 0{\cdot}5\mu}}{1+\mu} + (a_1 + a_2\sqrt{\mu} + a_3\mu)\sqrt{\mu}\left\{a_4\left(\frac{1}{\sqrt{n}} - 1\right) + a_5(n-1) + a_6(\sqrt{n} - 1)\right\}\frac{1}{\sqrt{n}} \tag{13}$$

$$R^{\mathrm{opt}} = \sqrt{\frac{2}{\mu}}(1+\mu) + (a_1 + a_2\sqrt{\mu} + a_3\mu)\sqrt{\frac{1}{\mu}}\left\{a_4\left(\frac{1}{\sqrt{n}} - 1\right) + a_5\left(\frac{1}{n} - 1\right) + a_6\left(\frac{1}{n\sqrt{n}} - 1\right)\right\} \tag{14}$$

The values of the coefficients in equations (11)–(14) are given in Table III. These values are obtained such that the mean square error is the minimum. Note that the first term in the above expressions denotes to the corresponding optimum value for the single TMD taken from Reference 5. A comparison of the optimum parameters obtained by numerical searching technique and above expressions is shown in Figures 3 and 4. It is observed from these figures that there is a good agreement between the optimum parameters of the MTMD system by two approaches. The maximum error for any value for ξ_T^{opt}, β^{opt}, f^{opt} and R^{opt} is observed to be 5·77,

Table III. Values of various coefficients in the explicit expression for optimum parameters

Coefficients	Corresponding value			
	ξ_T^{opt}	β^{opt}	f^{opt}	R^{opt}
a_1	0·5474	0·42113	− 0·00241	0·2985
a_2	0·1038	0·04479	0·72152	− 0·0078
a_3	− 0·4522	− 0·38909	− 0·43970	0·2355
a_4	0·7604	− 0·73518	− 0·66385	− 0·0442
a_5	0·3916	− 0·11866	− 0·01138	0·6265
a_6	0·0403	4·86139	0·99522	0·4789

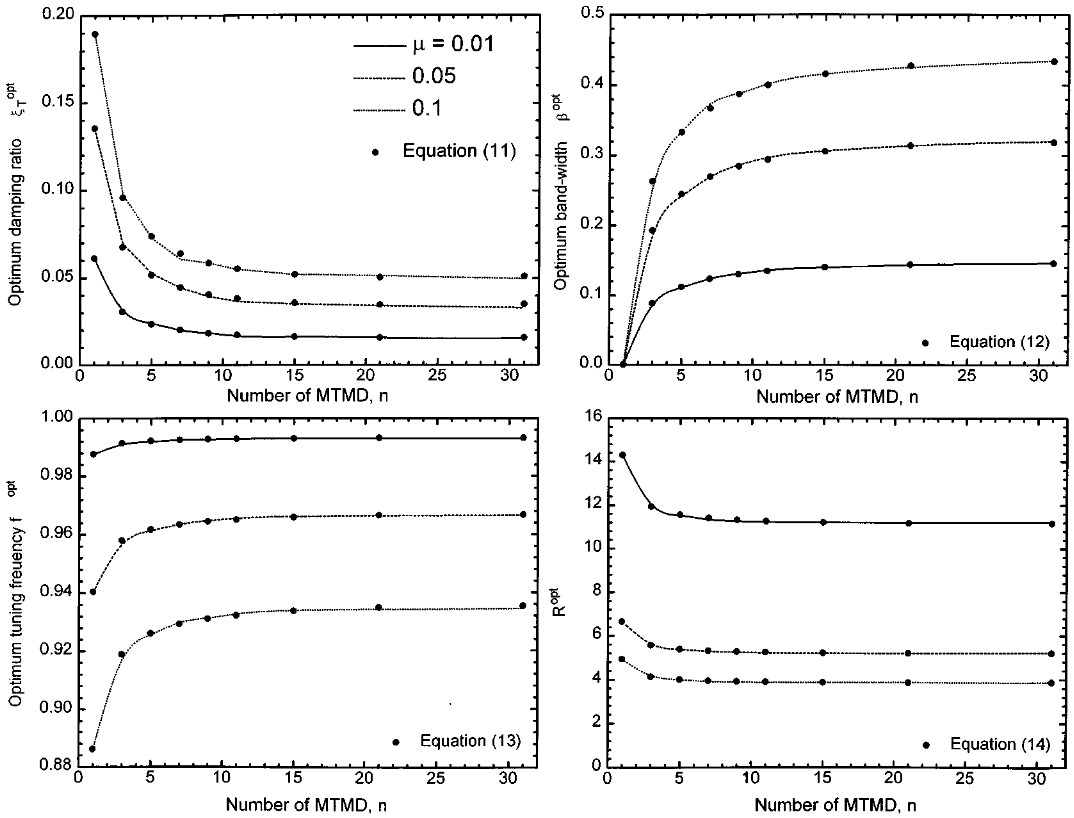

Figure 3. Comparison of optimum parameters by numerical searching technique and explicit expressions against the number of the MTMD

3.73, 0·21 and 1·45 percent, respectively. The magnitude of error for optimum damping ratio is relatively more. This is due to fact that the optimum damping ratio of the MTMD system is sufficiently low. From these figures, it is also observed that with the increase of number of MTMD the optimum damping ratio and displacement of the main system decreases whereas the optimum band-width and tuning frequency increases. On the other hand, optimum damping ratio and band-width increases and tuning frequency and corresponding displacement of the main system decreases with the increase of the mass ratio of the MTMD system.

The explicit formulas obtained here are based on the mass ratio not greater than 0·1 and number of MTMD in the range of 1–31. This covers the entire range of practical applications of the MTMD system. The use of proposed expressions beyond the above range may produce relatively more error. However, the same technique can be applied to find out the corresponding expressions for the other ranges of mass ratios and number of MTMD. In addition, the study is restricted for the optimum parameters of a main system subjected to harmonic-base acceleration. There is also a need for obtaining the optimum parameters for various combinations of response and excitation.

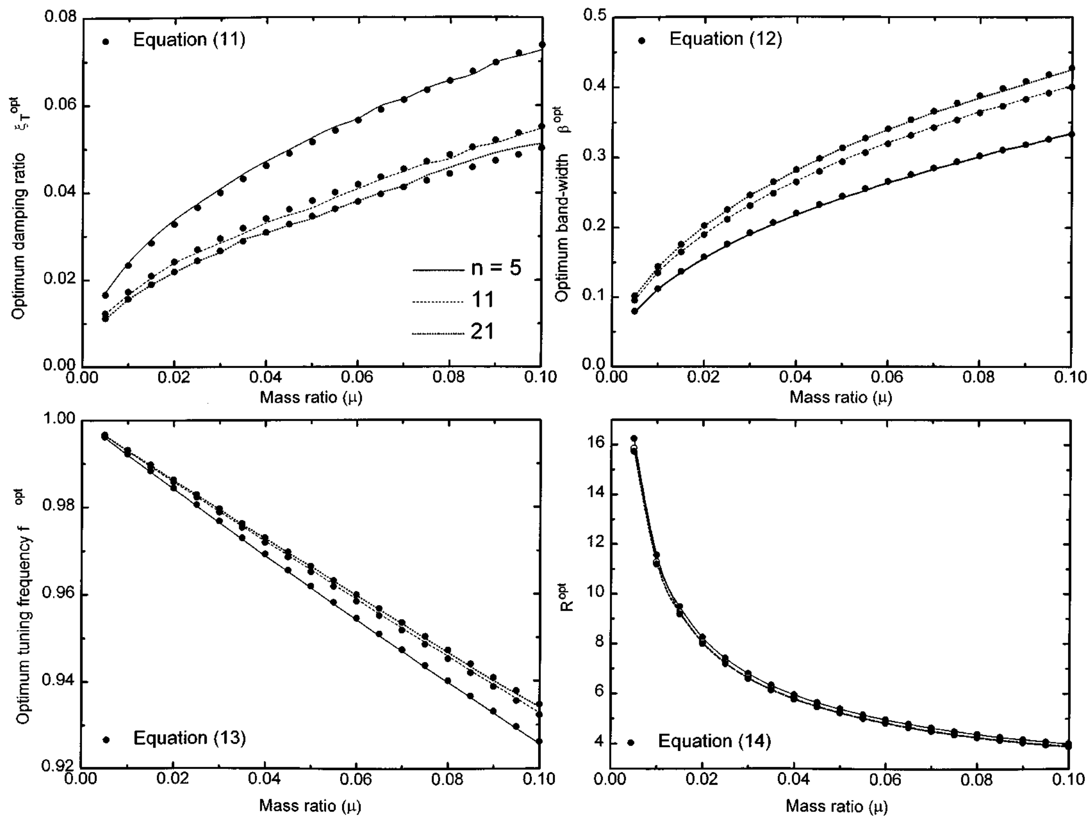

Figure 4. Comparison of optimum parameters by numerical searching technique and explicit expressions for different mass ratios of the MTMD

CONCLUSION

Using numerical searching technique the optimum parameters of the MTMD system for undamped main system are investigated. The criterion selected for the optimality is the minimization of the steady-state harmonic displacement of the main system to harmonic-base acceleration. A curve-fitting technique is then applied to find out the closed-form expressions for the optimum parameters of the MTMD system. Closed-form expressions has shown a good agreement with the corresponding optimum parameters obtained by numerical searching technique. Numerical study of the optimum parameters shown that (i) the optimum damping ratio of MTMD system decreases with the increase of the number of MTMD and increases with the increase of mass ratio, (ii) optimum band-width of the MTMD system increases with the increase of both the mass and number of MTMD and (iii) optimum tuning frequency increases with the increase of the number of MTMD and decreases with the increases of the mass ratio.

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